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SHORTER COMMUNICATIONS

HEAT TRANSFER TO NON-NEWTONIAN FLUIDS IN LAMINAR FLOW THROUGH CONCENTRIC ANNULI

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NOMENCLATURE

a_i, b_i, c_i,	series coefficients of problems A, B and C respectively;
A_i, B_i, Γ_i,	eigenfunctions of problems A, B, and C respectively;
$\alpha_i, \beta_i, \gamma_i$,	eigenvalues of problems A, B, and C respectively;
D_e,	hydraulic radius, $2R(1 - \kappa)$;
h,	heat transfer coefficient;
k,	thermal conductivity;
m, n	parameters of the power-law model;
Nu,	Nusselt number, hDe/k;
q,	flux of energy at the inner surface;
r,	radial position;
R,	radius of outer cylinder;
v_z,	axial velocity;
$V(n, \kappa, \zeta)$,	dimensionless velocity;
z,	axial distance;
$\Gamma(x)$,	gamma function;
ζ,	dimensionless radial position, r/R;
θ,	dimensionless temperature;
κ,	ratio of radius of inner cylinder to radius of outer cylinder;
λ,	dimensionless radial distance at which the velocity is a maximum;
ξ,	dimensionless axial distance, $zk/R^2 c_p v_{avg}$;
τ_{ij},	stress tensor.

Subscripts

e,	entrance condition;
wi,	inner surface condition;
avg,	bulk or cup-mixing.

1. INTRODUCTION

SEVERAL recent studies [1]–[4] have been devoted to problems of heat transfer to Newtonian fluids in laminar flow through annuli. Problems of heat transfer to non-Newtonian fluids in annuli have not, however, been extensively treated. Tien [5] and Skelland [6] have considered the limiting case of flow between parallel plates. Ziegenhagen [7] has obtained an approximate expression for the higher eigenvalues of the problem of constant heat flux at the inner surface which can be applied to non-Newtonian fluids and Trefethen [8] has presented Nusselt numbers for the fully developed temperature region assuming plug flow.

Heat transfer to non-Newtonian fluids in the thermal entrance region of annuli is considered in this note. In all cases the fluid enters the heat transfer region with a fully developed laminar velocity profile and with constant temperature T_e. Results are presented for the boundary conditions listed below.

A. Constant flux at the inner surface, outer surface insulated.
B. Step change in temperature at the inner surface, outer surface insulated.
C. Step change in temperature at the inner surface, outer surface maintained at T_e.

Inner–outer radius ratios of 0·2 and 0·5 have been studied for each of the cases listed.

The power-law model of non-Newtonian behavior has been assumed in all cases. For flows with velocity variation in only the radial direction the model is,

$$\tau_{rz} = -m\left|\frac{dv_z}{dr}\right|^{n-1}\frac{dv_z}{dr},$$

where m and n are constants which must be determined for the particular fluid in question. Experimental data of McEachern [9] indicates that the model predicts pressure drop vs. flow rate data reasonably well if attention is paid to the range of shear stress over which the parameters are evaluated.

The fully developed velocity profile predicted by the power-law model for steady laminar flow in an annulus has been determined by Fredrickson and Bird [10] and is given by:

$$\left.\begin{aligned} V(n,\kappa,\zeta) &= \frac{v_{\max}/v_{\mathrm{avg}}}{\int_\kappa^\lambda \left(\frac{\lambda^2}{\zeta}-\zeta\right)^{1/n}\mathrm{d}\zeta}\int_\kappa^\zeta\left(\frac{\lambda^2}{\zeta}-\zeta\right)^{1/n}\mathrm{d}\zeta, \\ &\qquad\qquad \kappa \leqslant \zeta \leqslant \lambda \\ V(n,\kappa,\zeta) &= \frac{v_{\max}/v_{\mathrm{avg}}}{\int_\kappa^\lambda \left(\frac{\lambda^2}{\zeta}-\zeta\right)^{1/n}\mathrm{d}\zeta}\int_\lambda^\zeta\left(\zeta-\frac{\lambda^2}{\zeta}\right)^{1/n}\mathrm{d}\zeta, \\ &\qquad\qquad \lambda \leqslant \zeta \leqslant 1. \end{aligned}\right\} \quad (1)$$

The two expressions must be the same at $\zeta = \lambda$, the radial position at which the velocity is a maximum, and this may be used to determine λ.

2. EXPRESSIONS FOR THE NUSSELT NUMBERS

The energy equation for a fluid in steady laminar flow with constant properties and with the assumptions of negligible viscous dissipation and axial conduction may be written,

$$V(n,\kappa,\zeta)\frac{\partial\theta}{\partial\xi} = \frac{1}{\zeta}\frac{\partial}{\partial\zeta}\left(\zeta\frac{\partial\theta}{\partial\zeta}\right), \quad (2)$$

where θ is defined by

$$\theta = \frac{T - T_e}{qR/k} \quad \text{for problem A}$$

and by

$$\theta = \frac{T - T_{wi}}{T_e - T_{wi}} \quad \text{for problems B and C.}$$

The boundary conditions for the various cases are:

		Problem A	Problem B	Problem C
$\xi \leqslant 0,$	$\kappa \leqslant \zeta \leqslant 1$	$\theta = 0$	$\theta = 1$	$\theta = 1$
$\xi > 0,$	$\zeta = \kappa$	$\partial\theta/\partial\zeta = 1$	$\theta = 0$	$\theta = 0$
$\xi > 0,$	$\zeta = 1$	$\partial\theta/\partial\zeta = 0$	$\partial\theta/\partial\zeta = 0$	$\theta = 1$

The solutions, which may be found by separation of variables, may be written;

$$\theta = \sum_{i=1}^{\infty} a_i A_i(\zeta)\exp(-\alpha_i^2\xi) + \frac{2\kappa}{1-\kappa^2}\xi + G(\zeta) \quad \text{Problem A}$$

$$\theta = \sum_{i=1}^{\infty} b_i B_i(\zeta)\exp(-\beta_i^2\xi) \quad \text{Problem B}$$

$$\theta = \sum_{i=1}^{\infty} c_i \Gamma_i(\zeta)\exp(-\gamma_i^2\xi) + 1 - \frac{\ln\zeta}{\ln\kappa} \quad \text{Problem C}$$

The Nusselt number at the inner surface is defined by,

$$Nu = \frac{hD_e}{k} = \frac{-\partial T/\partial r|_{r=\kappa R} D_e}{T_{wi} - T_{\mathrm{avg}}} \quad (3)$$

Converting to dimensionless quantities and substituting the above solutions yields;

$$Nu = \frac{2(1-\kappa)}{\sum_{i=1}^{\infty} a_i A_i(\kappa)\exp(-\alpha_i^2\xi) + G(\kappa)}, \quad \text{Problem A}$$

$$Nu = \frac{(1-\kappa)(1-\kappa^2)\sum_{i=1}^{\infty} b_i B_i'(\kappa)\exp(-\beta_i^2\xi)}{\kappa\sum_{i=1}^{\infty}\frac{b_i}{\beta_i^2}B_i'(\kappa)\exp(-\beta_i^2\xi)}, \quad \text{Problem B}$$

and for problem C,

$$Nu = \frac{(1-\kappa)(1-\kappa^2)\left[\sum_{i=1}^{\infty} c_i\Gamma_i'(\kappa)\exp(-\gamma_i^2\xi) - \frac{1}{\kappa\ln\kappa}\right]}{\sum_{i=1}^{\infty}[\Gamma_i'(1) - \kappa\Gamma_i'(\kappa)]\frac{c_i}{\gamma_i^2}\exp(-\gamma_i^2\xi) + \int_\kappa^1 V(n,\kappa,\zeta)\frac{\ln\zeta}{\ln\kappa}\zeta\,\mathrm{d}\zeta - \frac{1-\kappa^2}{2}}$$

The Nusselt number at the outer surface for problem C may be expressed as

$$Nu_0 = \frac{(1-\kappa)(1-\kappa^2)\left[\sum_{i=1}^{\infty} c_i\Gamma_i'(1)\exp(-\gamma_i^2\xi) - \frac{1}{\ln\kappa}\right]}{\sum_{i=1}^{\infty} c_i\exp(-\gamma_i^2\xi)\frac{1}{\gamma_i^2}[\Gamma_i'(1) - \kappa\Gamma_i'(\kappa)] + \int_\kappa^1 V(n,\kappa,\zeta)\frac{\ln\zeta}{\ln\kappa}\zeta\,\mathrm{d}\zeta}.$$

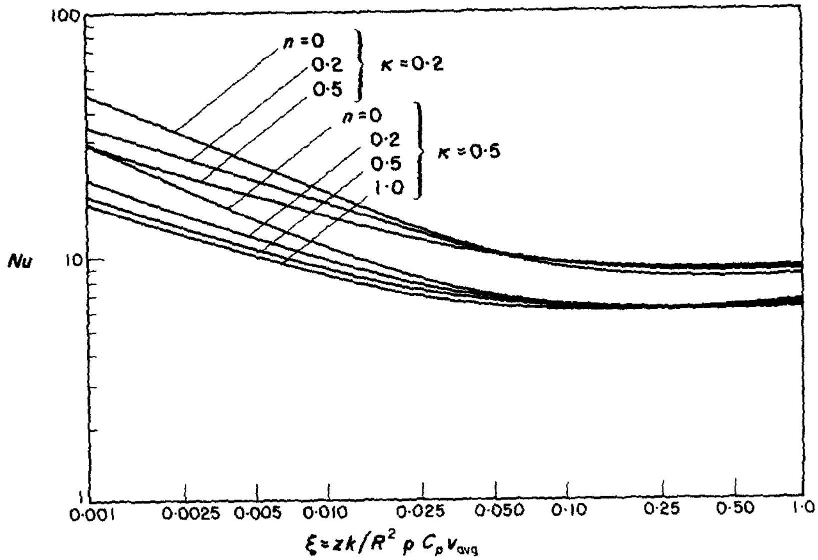

FIG. 1. Nusselt numbers for problem A.

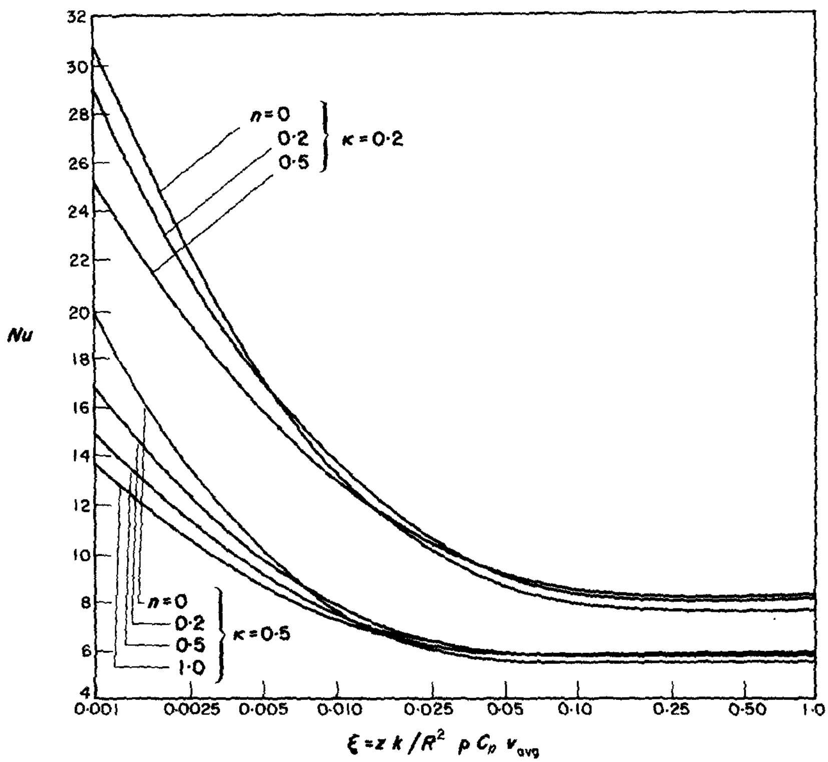

FIG. 2. Nusselt numbers for problem B.

The first twelve values of the quantities needed have been calculated and the resulting Nusselt numbers are shown plotted vs. axial distance in Figs. 1–3. The curves for $n = 1$ were taken from [1] and [2].

The Nusselt numbers at the inner surface for problem C are not shown. They are identical with those of problem B near the entrance but approach values some 12 per cent lower than those of problem B for $\kappa = 0{\cdot}2$ and 15 per cent lower for $\kappa = 0{\cdot}5$ in the fully developed temperature range.

Most of the eigenvalues in the above problems were determined by the iterative method of Berry and de Prima [11], [1], though some of the higher values were determined by the WKB asymptotic method. When $n = 0$ (plug flow) the solutions were expressed in terms of Bessel functions and the eigenvalues determined from tabulated values of the functions.

Very near the entrance a large number of terms are needed in the series expressions to accurately determine the Nusselt numbers. It is desirable, therefore, to have a simple approximate solution for this region. Such a solution is obtainable from the familiar Leveque procedure which is based on the concept that near the entrance the heat has penetrated only a very short distance into the fluid. The solutions in terms of a similarity variable may be found in, for example, Bird *et al.* [12].

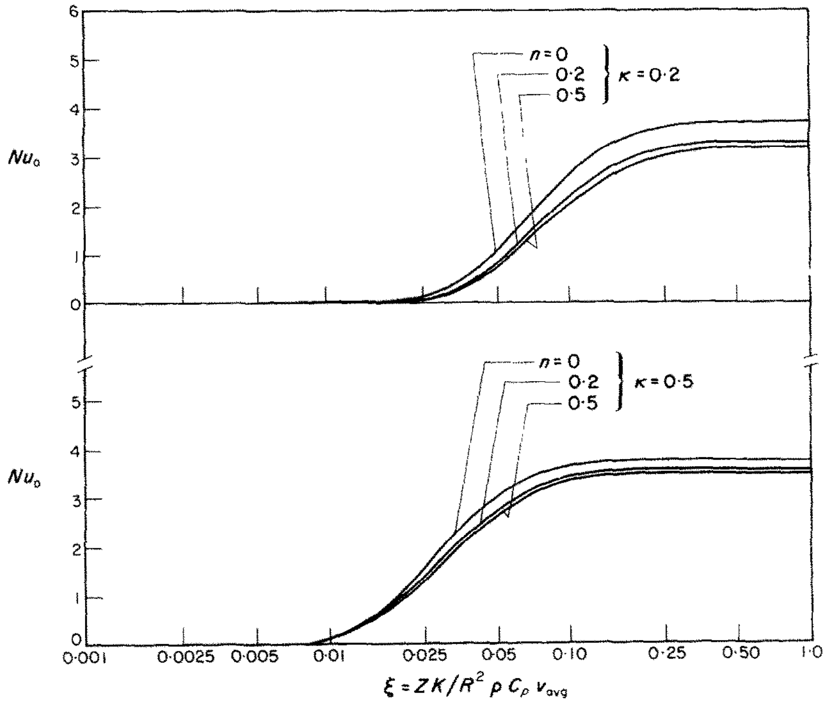

FIG. 3. Nusselt numbers at outer wall for problem C.

3. DISCUSSION

The agreement between the complete solutions and those valid near the inlet is quite satisfactory for the $n = 0$ and $n = 0{\cdot}5$ cases and it is recommended that the approximate solutions be used for obtaining Nusselt numbers for values of ξ less than 0·0006.

Comparing the two solutions when $n = 0{\cdot}2$ is difficult. The linearized velocity profile used in the approximate solution is valid over only a very small radial distance, thus the approximate solution is valid only a short distance from the thermal entrance and a very large number of eigenvalues and eigenfunctions are needed to safely extrapolate the complete solution to this region. Despite these difficulties the error between the two solutions at $\xi = 0.001$ is not excessive, ranging from 4·8 per cent to 8·1 per cent (problem B, $\kappa = 0.5$) with the approximate solution predicting the higher value. These errors persist in the range $0.0001 < \xi < 0.001$.

The Nusselt numbers for the fully developed temperature profiles of problem A are shown in Fig. 4. In general, the radius ratio is a more significant parameter than the power law parameter, n. It should be noted that for κ values greater than 0·5 the Nusselt numbers will rapidly approach those of parallel plates.

As can be seen from Fig. 4 the non-Newtonian velocity

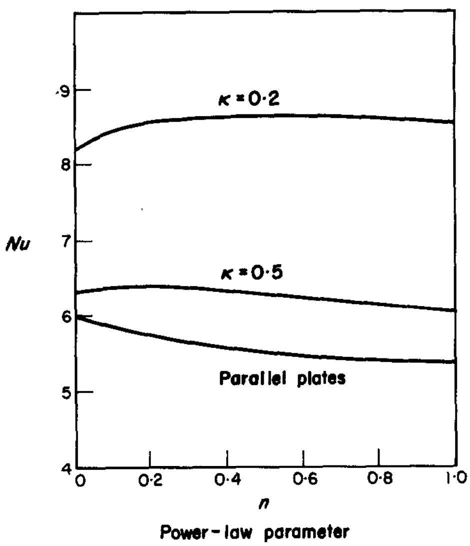

FIG. 4. Nusselt numbers for the fully developed temperature region, problem A.

profile with $0 < n < 1$ can result in higher rates of convective heat transfer than either plug or Newtonian flow. This is because the annular non-Newtonian velocity profile possesses, to varying degrees, the high velocity of the Newtonian fluid in the middle portions of the conduit and also, because of the very steep velocity gradient, the high velocity of plug flow near the inner surface.

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